INVESTIGATING STUDENTS' GEOMETRICAL PROOFS THROUGH THE LENS OF STUDENTS' DEFINITIONS

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We present the second stage of a study within the context of geometry, whose aim is to investigate relationships between and influence of visualization, the concept images of students concerning geometrical concepts and their definition, and students' ability to prove. We focus on links between the understanding of the definition's role in concluding the geometrical concept attributes and proofs that deal with these attributes. We exemplify this stage in our research, by means of examples, which reveal that the difficulties students have in understanding the geometric concepts' definitions affect the understanding of the proving process and hence the ability to prove.

INTRODUCTION AND BACKGROUND

The research reported here is part of a larger study aimed at investigating: 1) the effect of visualization and of students' concept images on students' construction of geometrical concepts and their definitions; 2) the effect of definitions on students' ability to prove in geometry, and 3. the effect of visualization and concept formation difficulties on students' ability to prove in geometry. At the previous PME conference we reported on findings from investigating point 1 (Haj-Yahya & Hershkowitz, 2013). In this research report we focus mainly on findings concerning point 2.

The research literature includes many studies on the meaning of proof for students (e.g. Fischbein & Kedem, 1982) and on their ability to prove in geometry (e.g. Martin, McCrone, Bower & Dindyal, 2005). But little research was done concerning the effect of definitions on proving in geometry. Moore (1994) investigated the ability to prove concerning non-geometrical concepts. His participants were university students. He found that the superficial understanding of concept definitions and images prevented students from starting proofs and from seeing the overall structure of a proof. Edwards and Ward (2004) found that students have a tendency to rely on their concept images instead of the related concepts. Again their research context was non-geometrical concepts. It is especially surprising that there is so little research attempting to investigate the relationship between definition and proving in geometry, while school curricula in many countries dedicate most of the time devoted to learning geometry in high school to the subject of definitions and proofs. The present research attempts to fill this gap.

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THE STUDY (SECOND STAGE)

At the previous PME conference, we exemplified our findings concerning point 1 above, by means of paradigmatic examples, which reveal students' visual and verbal processes related to construction of geometric figures and inclusion relationships between groups of figures and their attributes. Our results confirmed known findings, for example that the position of a shape affects its identification and the related inclusion relationships (e.g., Hershkowitz, 1989) and also pointed to findings in a new direction, such as the effect of the question's representation on students' responses concerning the inclusion relationships. Here we focus on the role of definitions in processes of geometrical proving.

Population

The participants are 90 students from a regional high school in an Arab community in the centre of Israel, all of whom participated in stage 1 of the research. They learn geometry with three different teachers in three parallel classes, which are considered to be at the highest mathematical level among the seven parallel classes in this school. All teachers have a first degree in mathematics from the universities in the country and more than ten years of experience in teaching mathematics.

Methodology

The main research tools of the three-stage research include three questionnaires, one for each stage. The questionnaires were administered at time intervals sufficient for analyzing the results of each questionnaire and use its findings in the design of semi-structured interviews with about 10% of the study participants, and in the design of the next stage questionnaire for the whole population. The questionnaire used in this 2nd stage of the study deals with defining and proving (related to quadrilaterals). After administering the questionnaire and analyzing its results, nine students were interviewed.

In the tasks of this questionnaire the students were asked to "reflect on other students' answers". During such reflection, students had opportunities to use *critical thinking*; they test the proof made by the "other student". Also, while students are required to explain their responses, they uncover some of their views and knowledge regarding proving processes. Detailed analyses of a few questionnaire tasks and of students' responses are given in the next section.

DATA COLLECTION, ANALYSIS AND FINDINGS

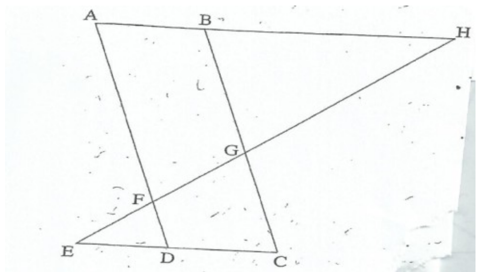
The data *of the second stage* were collected in 2013, while the participants were in the grade 11. Questionnaire 2 includes 5 tasks and was administered at the end of the first semester. In the following, we focus on and analyse data from the participants' responses to three tasks in this questionnaire.

The Trapezium Task (Figure 1): In the Trapezium Task we provided an insufficient proof, given supposedly by a student called Ramie. The students were asked to check the proof's correctness and to explain their responses. The aim of this task is to examine whether the students pay attention to a missing step in a given proof. This task was designed because while analysing the first questionnaire we found that only 27% of the participants gave a correct definition of trapezium. In our curriculum, a trapezium is defined as a quadrilateral with exactly one pair of parallel sides.

Trapezium: is a quadrilateral with only one pair of parallel sides.

Problem: *ABCD* is a given parallelogram, E and H are on the continuation of sides CD and AB, respectively. EH intersects AD and BC at points F and G, respectively.

Prove that *ABGF* is a trapezium.



Here is Ramie's proof: ***ABCD is a parallelogram, therefore AD and BC are parallel. BG is part of BC and AF is part of AD, hence AF and BG are parallel (parts of parallel sides). We found a pair of opposite parallel sides, therefore ABGF is a trapezium.***

Did Ramie give a correct and complete proof? Explain your response!

Figure 1: The Trapezium Task

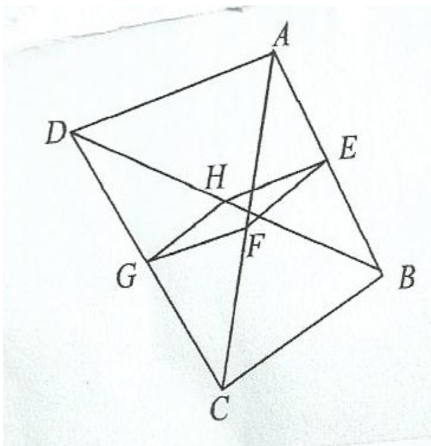
Student's claim \ Explanation	No explanation	Should prove that the other pair of sides intersect	The shape could be: parallelogram , rhombus…	According to the drawing the other two sides intersect	Use insufficient definition of the trapezium	Use incorrect definition of the trapezium	Total
No claim	2 (2.2%)	0	0	0	1 (1.1%)	1 (1.1%)	4 (4.4%)
Correct & complete proof	3 (3.3%)	12 (13.3%)	1 (1.1%)	1 (1.1%)	37 (41.1%)	3 (3.3%)	57 (63.3%)
Incomplete proof	1 (1.1%)	14 (15.5%)	10 (11.1%)	0	3 (3.3%)	1 (1.1%)	29 (32.2%)
Total	5 (6.6%)	26 (28.9%)	11 (12.2%)	1 (1.1%)	41 (45.5%)	5 (5.5%)	90 (100%)

Table 1: Participants' responses to the Trapezium Task

Table 1 shows that 63% of the participants claim that the proof is correct & complete, and yet the majority (65%) of them based their justifications on an insufficient definition for trapezium. E.g. student a13 wrote: *Ramie's proof is correct, he found and proved that there is a pair of parallel sides*. It is very interesting to see that there are 12 students (13%) who claimed that the proof is correct, although they wrote that Ramie should prove that the other sides are intersect, they paid attention to the proof incompleteness, but their final answer was not consistent with their argument. Only 32% of the students claimed that the proof is incomplete, whereas about half of them explained explicitly that Ramie should prove that the other pair of sides intersect; e.g. student b43 wrote: *not correct because it is incomplete (proof process), he should prove that the other pair are not parallel, AB is not parallel to FG because they intersect in point H*. About one third of the students in this category explained that without completing the proof, the shape could be a different one (not a trapezium). E.g. b41claimed that when we accepted this proof, parallelogram considered as trapezium because there is one pair parallel sides in parallelogram, he wrote: *No, Ramie's proof is correct but not complete. This definition fits other concepts, for example it fits a parallelogram.* We may conclude that here we have evidence that many students are not consistent concerning incomplete proof although the correct definition given at the top of the task states explicitly that there is only one pair of parallel sides.

The Parallelogram Task (Figure 2): This task deals with a proof that a certain quadrilateral is a parallelogram. This may be done by showing that each pair of opposite sides are parallel, or that each pair of opposite sides are equal, or that there is one pair of opposite sides which are equal and parallel. In each case the proof is sufficient. We represented a *non-economical proof*. This task was inserted into this stage, because after analyzing the first questionnaire we realized that students have a tendency to give a *non-economical definition* for the parallelogram.

ABCD is quadrilateral, E is in the middle of AB, G in the middle of DC, F in the middle of AC and H in the middle of BD.
Prove that HEFG is parallelogram.
Ahmed wrote the following proof:



We can see that FE and GH are mid-segments in triangles ABC and DBC, respectively, thus because of the mid-segment attributes we can conclude that $GH = FE = \frac{1}{2}BC$ and GH is parallel to FE.

Remains to prove that the other sides are equal and parallel. HE and GF are mid-segments in triangles ADB and ADC, respectively. Therefore we can conclude that $GF = HE = \frac{1}{2}AD$ and GF is parallel to HE. We have proved that there are two pairs of opposite sides equal and parallel, therefore the shape is parallelogram.

Do we need all the steps Ahmad made? If so explain why, if not what steps can be omitted?

Figure 2: The Parallelogram Task

The results in Table 2 indicate that the tendency to give a non-economical definition for a parallelogram appears to have an influence on the process of proving, and many students adopt the non-*economical proof*: 57% of all students claim that all the steps are necessary.

Only 37% of the participants wrote that there are superfluous steps in the proof, and 75% among these students explained their response by using an economical definition; e.g. a16 wrote: *It is not necessary to do all the steps. Ahmed could only prove that one pair of sides are equal and parallel*.

Explanation / Student's claim	Didn't explain	Used an economical definition	Used non-economical definition	Used insufficient definition	Tautology	Wrote unrelated things	Total
Didn't claim	4 (4.4%)	2 (2.2%)	0	0	0	0	6 (6.6%)
All steps are necessary	21 (23.3%)	4 (4.4%)	4 (4.4%)	0	4 (4.4%)	18 (20%)	51 (56.6%)
There are superfluous steps	3 (3.3%)	25 (27.7%)	2 (2.2%)	1 (1.1%)	0	2 (2.2%)	33 (36.6%)
Total	28 (30.8%)	31 (34.1%)	6 (6.6%)	1 (1.1%)	4 (4.4%)	20 (22%)	90 (100%)

Table 2: Students' responses to the Parallelogram Task

The issues in this task were investigated by interviews as well. Here is an episode from one of the interviews:

(I – interviewer; A – Aseel, a student: discussing the Parallelogram Task)

1 I: Among two students from your class, one proved only that each two opposite sides are parallel. The other student proved only that each two opposite sides are equal.

2 A: O.K.

3 I: Which answer is correct? Are both of them correct? Is one of them correct? Is any answer correct?

4 A: Both are wrong, because in the parallelogram each pair of opposite sides are parallel and equal.

5 I: So, which answer do you prefer?

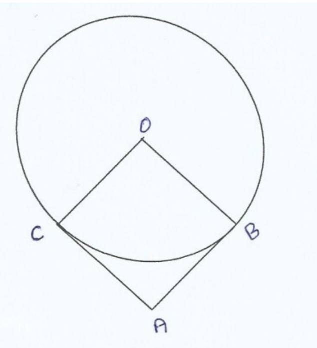
6 A: The first in which the student proves that each pair of opposite sides are parallel.

7 I: Why?

8 A: We call it parallelogram, parallel, the word parallel must be.

Aseel does not understand the "mathematical agreement" that a definition has to be minimal and that there are often equivalent definitions. In this episode Aseel (4) shows that like another 57% of the students she thinks that "all steps are necessary". Her way of expressing it indicates that she is confused between the set of all attributes a parallelogram has, and a minimal set of attributes sufficient for the definition of a parallelogram. She does not attend to or does not understand the concept of economical definition and hence not the concept of economical proof either. In addition, the attribute ‘parallel’, which is part of the figure's name affects Aseel's definitions and therefore affects her preference for proving.

Rectangle Or Not Task: This task (See Figure 3) had three subtasks, but here we will relate to subtask b only. Our aim here is to investigate if and in what way understanding (or not) the inclusion relationships between groups of quadrilaterals is expressed in proving. Especially we want to know whether the students will use the rectangle and kite definitions or not. In the analysis of the first questionnaire we found that only 7% correctly identified the square as a kite and about 17% identified the square as a rectangle.



Definition: A rectangle is a parallelogram with one right angle.

Problem: There is a circle with center O, OB=OC are two radii. They are *perpendicular*. From point A outside the circle we draw two tangents to the circle: AB and AC.

Is *ABOC* a rectangle? If not which quadrilateral it is? Prove your answer!

Mohamed says*: **In the quadrilateral ABOC there are three right angles. In addition OB=OC (the radii are equal), therefore all 4 sides are equal. Hence the quadrilateral ABOC is a square and can't be a rectangle or a kite, because in a rectangle and a kite not all sides are equal.***

Is Mohamed's proof correct? If not, explain your response!

Figure 3: Rectangle or not Task.

The main findings from Table 3 are: Only a third of the students claimed that Mohamed’s proof was wrong. But only 27% of these use a correct definition of a kite or a rectangle, or correctly identified the inclusion relationships between the squares and rectangles and between the squares and kites; e.g. c10 writes: *because all 4 sides are equal and all angles are right angles and the square is a kite and also a rectangle*. About half of the students claimed that Mohamed's proof is correct and did not relate to the fact that the square has all the critical attributes of the rectangle and kite concepts. Whereas 57% among them didn’t explain their responses (they were not asked to do it) and about 30% among them referred only to the square. E.g. c8 writes: *"right, according to what he proved the constructed shape is a square and not rectangle because he proved that there are 4 equal sides and 4 right angles"*. Again we have evidence that the difficulties in understanding the inclusion relationships among the groups of quadrilaterals and their attributes influence the ways the students deal with and evaluate proofs.

Explanation / Student's claim	Didn't explain	Correctly used definitions of kite and rectangle	Correctly used the inclusion relationships	Claimed that Mohamed should prove that the other two sides are equal	Claimed that the square is a kite and not a rectangle	Only claimed that the shape is a square	Wrote unrelated things	Total
Didn't decide	8	0	0	2	0	0	0	10
	(8.9%)			(2.2%)				(11.1%)
Mohamed's proof is correct	25	0	1	0	0	13	5	44
	(27.7%)		(1.1%)			(14.4%)	(5.5%)	(48.9%)
Mohamed's proof is incorrect	1	4	4	8	2	4	7	30
	(1.1%)	(4.4%)	(4.4%)	(8.9%)	(2.2%)	(4.4%)	(7.7%)	(33.3%)
Mohamed's proof is partially correct	0	0	1	1	0	4	0	6
			(1.1%)	(1.1%)		(4.4%)		(6.6%)
Total	34	4	6	11	2	21	12	90
	(37.7%)	(4.4%)	(6.6%)	(12.2%)	(2.2%)	(23.3%)	(13.3%)	(100%)

Table 3: Students' responses to the Rectangle or not Task

CONCLUDING REMARKS

We can see a general and clear tendency: Student's difficulties in understanding the definitions of geometrical concepts affect these students' proof processes. These difficulties affect proof processes wherever these processes rely on the definitions. This tendency is in agreement with Knapp (2006). We can interpret some of these difficulties by the lack of students' understanding that a definition must on one hand not contain any superfluous information (see the Parallelogram Task), but must on the other hand contain a necessary and sufficient set of attributes. The other difficulties might be explained by the students' lack of understanding the two directions of inclusion relationships (see the third task): inclusion relationships between groups of quadrilaterals in one direction and the inclusion relationships of their attributes in the opposite direction (Hershkowitz et al., 1990). It is worth to note that in spite of what we claimed above, there are cases in which students are not attentive to incomplete proof although the correct definition is given as in the findings of the Trapezium Task.

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